Milne's rationality conjecture

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Abstract

We prove Milne's rationality conjecture for reductions of Hodge classes on abelian varieties: The cohomological realizations of such classes pair rationally with Lefschetz classes (that is, classes in the algebra generated by divisor classes) in complementary codimension. The proof proceeds by establishing a dichotomy for reductions of Weil classes (an expected consequence of the rationality conjecture): Either they are orthogonal to all Lefschetz classes or are themselves Lefschetz classes. The key new input are lifting results of Kisin–Zhou for Hodge-type Shimura varieties. This is a cleaned up version of a preprint produced by OpenAI [OAI26:Milne]

1 The theorem and the argument

An abelian variety can acquire new divisors after good reduction. Milne's rationality conjecture asks whether a rational Hodge class on the original variety has rational intersection numbers with these new divisors, with the same value in every cohomological realization. Here the relevant intersections are with products of divisors of complementary codimension.

The key new input into the proof is provided by Kisin–Zhou's lifting results for Hodge-type Shimura varieties [KZ25, Section 2.2]. They allow one to change a characteristic-zero lift, after moving within the isogeny class of its reduction, while retaining prescribed tensors. We use both their CM lifting theorem and their construction of lifts realizing an admissible Hodge filtration. These results make it possible to carry out the simultaneous-lifting strategy proposed by Milne [Mil08, Question 10.6 and Proposition 10.7].

Fix $\overline{\mathbf{Q}} \subset \mathbf{C}$, a prime p, and a p-adic place w of $\overline{\mathbf{Q}}$. Put

$$k = \overline{\mathbf{F}}_p, \qquad K_0 = W(k)[1/p], \qquad C_w = \widehat{\overline{\mathbf{Q}}_p},$$

with compatible embeddings. Good reduction is understood after descent to a number field and finite extension at w. If γ is a rational Hodge class, Deligne's absolute-Hodge theorem and the Blasius–Wintenberger comparison theorem give its specialized realizations $\gamma_{0,v}$, for $v = \ell \neq p$ and $v = p$ [Del82, Bla94, Moo98]. In the crystalline realization we extend coefficients from K_0 to C_w.

Theorem 1.1 (Milne's rationality conjecture)**.** *Let $A/\overline{\mathbf{Q}}$ be an abelian variety of dimension d with good reduction A_0/k, and let*

$$\gamma \in H^{2r}(A(\mathbf{C}), \mathbf{Q}(r))$$

be a rational Hodge class. For every rational Lefschetz class δ of codimension $d-r$ on A_0, there is $q \in \mathbf{Q}$ such that

$$\mathrm{Tr}_v(\gamma_{0,v}\,\delta_v) = q \qquad (v = \ell \neq p \text{ or } v = p).$$

Here a rational Lefschetz class is a rational polynomial in divisor classes; $\mathrm{Lef}^r(A_0)$ *denotes their space modulo numerical equivalence. All traces have their cycle-class normalization.*

Equivalently, there is a unique $\lambda_\gamma \in \mathrm{Lef}^r(A_0)$ *such that, simultaneously in every realization,*

$$\gamma_{0,v} = (\lambda_\gamma)_v + \gamma_{0,v}^{\perp}, \qquad \gamma_{0,v}^{\perp} \perp \mathrm{Lef}^{d-r}(A_0).$$

Thus the Lefschetz projections of the specialized Hodge class are the realizations of one rational class.

The proof reduces to a more concrete assertion about Weil classes. Let $Z/\overline{\mathbf{Q}}$ have good reduction Z_0 and an action of a CM field E, with $\mathrm{rk}_E\, H^1(Z, \mathbf{Q}) = 2s$, balanced Hodge multiplicities (s, s) at every embedding, and a polarization inducing complex conjugation on E. The associated space $W_E^s(Z)$ consists of the rational classes whose components, after splitting E, lie in the determinant lines of its eigenspaces; see Section 2.3. For $\gamma \in W_E^s(Z)$, either

$$\mathrm{Tr}_v(\gamma_{0,v}\, \delta_v) = 0 \quad \text{for every complementary Lefschetz class and every } v,$$

or there is a single $z \in \mathrm{Lef}^s(Z_0)$ whose realization is $\gamma_{0,v}$ for every v. Thus a nonzero Lefschetz pairing forces the specialized Weil class itself to be a simultaneous rational Lefschetz class. This proves a stronger form of the expected orthogonal decomposition first for reductions of Weil classes: each such class lies entirely in one of the two summands. The reductions through CM and Weil classes then establish rationality for arbitrary Hodge classes, and perfectness of the Lefschetz pairing gives their general orthogonal decomposition.

The proof is organized by the reductions that lead to this dichotomy.

1. *Pass to CM.* Kisin–Zhou's special-point lifting theorem, with a tensor-preserving marking of the special fiber, expresses the specialization of γ as a rational pullback of a specialized Hodge class on a CM abelian variety.

2. *Pass to Weil classes.* The Deligne–André–Milne reduction expresses a CM Hodge class as a sum of rational pullbacks of balanced Weil classes. The desired pairing property is preserved by these pullbacks.

3. *Extract an endomorphism.* Write θ for the polarization class and $\dagger$ for its Rosati involution. A nonzero Lefschetz pairing produces an invertible $u \in \mathrm{End}^0(Z_0)$ satisfying

$$u^\dagger = u, \qquad ue = \bar{e}u \quad (e \in E), \qquad \mathrm{Tr}_v(\gamma_{0,v} D_u^s \theta^{\dim Z - 2s}) \neq 0.$$

   Here $\bar{e}$ is CM conjugation and D_u is the rational divisor corresponding to the symmetric homomorphism obtained by composing the polarization with u.

4. *Lift simultaneously.* The common centralizer of E and u in the polarization similitude group has symplectic factors with one common multiplier. Its isocrystal satisfies the standard nonemptiness criterion for a balanced weakly admissible Hodge filtration. Kisin–Zhou's lifting construction realizes this filtration on an abelian variety, retaining both the Weil class and u.

5. *Use the Lefschetz pairing.* Over $\mathbf{C}$, the Weil-class space is an irreducible, multiplicity-one representation of $E^\times$. The rational projector onto the Lefschetz algebra therefore acts on it as either zero or the identity. The preserved nonzero pairing forces the latter. A rational divisor expression on the lift then specializes in every realization.

Section 2 carries out the CM and Weil reductions. Sections 3 and 4 construct the endomorphism and a filtration preserving it; these are the ingredients that allow the second application of Kisin–Zhou. Section 5 proves the dichotomy and the theorem. Appendix A gives the precise lifting statements used in both applications, including the quasi-isogenies that identify the specialized tensors.

1.1 Authorship

This document was produced by ChatGPT Astra 6 via questions, feedback and light editing from Keerthi Madapusi. Claude Opus 5.5 contributed part of the content of §4. Markdown file with the entire ChatGPT log is available here: https://keerthimadapusi.com/assets/ai-notes/milne-rationality/milne-rationality-chat.md

2 Reduction to balanced Weil classes

The two reductions in this section express a Hodge class as a sum of pullbacks from other abelian varieties. We first record that rationality of its Lefschetz pairings is preserved by these operations. Kisin–Zhou's CM lifting theorem then reduces the problem to CM Hodge classes, and the Deligne–André–Milne construction reduces these to Weil classes.

2.1 The rational Lefschetz algebra

For an abelian variety X/k, write $\mathrm{Lef}^r(X)$ for the rational span of products of r divisor classes, modulo numerical equivalence. Put $k_\ell = \mathbf{Q}_\ell$ for $\ell \neq p$ and $k_p = C_w$. Its cycle maps are injective and identify $\mathrm{Lef}^r(X) \otimes_{\mathbf{Q}} k_v$ with the divisor-generated subspaces of the corresponding cohomology. The same statements hold in rational Betti cohomology for a complex abelian variety. The intersection pairing

$$\mathrm{Lef}^r(X) \times \mathrm{Lef}^{\dim X - r}(X) \longrightarrow \mathbf{Q}$$

is perfect. Pullback, pushforward, and the Lefschetz operators preserve these subspaces [Mil99, Section 5]. We call the realizations of a single element of $\mathrm{Lef}^r(X)$ a *simultaneous rational Lefschetz class*.

We work with rational homomorphisms of abelian varieties. If $f \in \mathrm{Hom}^0(X, Y)$ and Nf is an actual homomorphism, then on cohomology of degree $2r$,

$$f^* = N^{-2r}(Nf)^*. \tag{1}$$

This convention is the same in every realization.

Definition 2.1. A tuple $\alpha = (\alpha_v)_v$ of classes of codimension r on X has the *rational pairing property* if, for every $\delta \in \mathrm{Lef}^{\dim X - r}(X)$, the values $\mathrm{Tr}_v(\alpha_v \delta_v)$ are the images of a single rational number.

Lemma 2.2. *The rational pairing property is preserved by rational linear combinations and by pullback along rational homomorphisms.*

Proof. Only pullback requires explanation. First let $f : X \to Y$ be an actual homomorphism. If $\delta \in \mathrm{Lef}^{\dim X - r}(X)$, then $f_* \delta_v$ belongs to the complementary Lefschetz subspace of Y in every realization. Its coordinates in a rational basis are rational and independent of v: pairing with

a rational dual basis computes them as the intersection numbers $\deg(\delta\, f^*\epsilon)$ on X. Thus $f_*\delta$ is one simultaneous rational Lefschetz class. The projection formula proves the assertion for $f^*\alpha$. Equation (1) gives the assertion for rational f; degrees outside the cohomological range are zero. □

Proposition 2.3 (Rationality and the orthogonal decomposition)**.** *Let $\alpha = (\alpha_v)_v$ be a tuple of classes of codimension r on an abelian variety X of dimension d. It has the rational pairing property if and only if there is a unique $\lambda \in \mathrm{Lef}^r(X)$ such that*

$$\alpha_v = \lambda_v + \alpha_v^{\perp}, \qquad \mathrm{Tr}_v(\alpha_v^{\perp}\delta_v) = 0 \quad \textit{for all } \delta \in \mathrm{Lef}^{d-r}(X)$$

in every realization. In this sense the decomposition into a Lefschetz part and an orthogonal part is defined over $\mathbf{Q}$ *and independent of the realization.*

Proof. Perfectness identifies $\mathrm{Lef}^r(X)$ with the rational dual of $\mathrm{Lef}^{d-r}(X)$. If α has the pairing property, its common rational functional on the latter space is represented by a unique $\lambda \in \mathrm{Lef}^r(X)$. Subtracting λ_v gives the asserted orthogonal remainder. Conversely such a decomposition computes every pairing as the rational intersection number $\deg(\lambda\delta)$. □

The projection is a formal consequence of rationality. The proof below first obtains its stronger, pure form for specialized Weil classes: either their projection is zero, or it is the class itself. Rational pullback preserves the pairing property, although it need not preserve this pure alternative. This is why the final theorem asserts a decomposition for general Hodge classes.

2.2 Transport to a CM point

Kisin–Zhou's CM lifting theorem lets us replace the original characteristic-zero lift while retaining its Hodge tensors on the special fiber. We use it in the following form, allowing a power of the original abelian variety.

Proposition 2.4 (CM transport)**.** *Let $A/\overline{\mathbf{Q}}$ have positive dimension and good reduction A_0, and let γ be a rational Hodge class on A. There are a positive integer n, a CM abelian variety $C/\overline{\mathbf{Q}}$ with good reduction, a quasi-isogeny $a : A_0^n \to C_0$, and a rational Hodge class κ on C such that*

$$(a \circ j)^*\kappa_{0,v} = \gamma_{0,v} \qquad \textit{for every } v,$$

where $j : A_0 \to A_0^n$ includes the first factor.

Proof. Use the Mumford–Tate Shimura datum of A and apply the CM alternative of Proposition A.1. The Hodge class is a tensor of that datum, with its Tate twist. The tensor-preserving quasi-isogeny gives the required equality in every realization. □

The power A^n allows the auxiliary Shimura construction in Appendix A; restriction to its first factor recovers the class on A_0.

2.3 Balanced Weil classes

Let $A/\mathbf{C}$ be a polarized abelian variety of dimension g, with an action of a CM field E such that Rosati induces complex conjugation on E. Suppose $V = H_1(A, \mathbf{Q})$ has rank $2s$ over E, where $s \geq 1$. Thus $g = s[E : \mathbf{Q}] \geq 2s$. Over a splitting field Ω of E, write V_τ for the eigenspace belonging to an embedding $\tau : E \hookrightarrow \Omega$. The Weil space is

$$W_E^s(A)_\Omega = \bigoplus_{\tau : E \hookrightarrow \Omega} \bigwedge^{2s} V_\tau^*(s) \subset H^{2s}(A, \Omega(s)).$$

Each summand is extended by zero on the other eigenspaces. We say that A is of *balanced Weil type* if its Hodge signatures are (s, s) at every embedding. In this case $W_E^s(A)$ consists of rational Hodge classes.

2.4 From CM Hodge classes to Weil classes

We use the Deligne–André reduction in the following form; compare [And92] and [Mil20, Theorem 1]. A short account of the construction records why good reduction is preserved.

Proposition 2.5. *Let $C/\overline{\mathbf{Q}}$ be of CM type, with good reduction at the chosen place. Every rational Hodge class κ of positive codimension r can be written*

$$\kappa = \sum_i f_i^* \eta_i,$$

where $f_i \in \mathrm{Hom}^0(C, Z_i)$ and η_i is a rational Weil class on Z_i. Each Z_i has good reduction and an action of a CM field E_i for which $H_1(Z_i, \mathbf{Q})$ has rank $2r$, all Hodge signatures are (r, r), and some polarization induces conjugation on E_i. The equality specializes in every realization.

Proof. Choose a CM algebra E' acting on C such that $V = H_1(C, \mathbf{Q})$ is free of rank one over E', and choose a finite Galois CM field F splitting E'. Write $V \otimes F = \bigoplus_{\rho \in S} V_\rho$, where $S = \mathrm{Hom}_{\mathbf{Q}\text{-alg}}(E', F)$ and each V_ρ is a line. The components of $\bigwedge^{2r} V^*$ are indexed by subsets $\Delta \subset S$ of size $2r$. If a rational Hodge class has a nonzero component at Δ, then every Galois conjugate of Δ contains exactly r lines of each Hodge type.

In the isogeny category form $C \otimes_{\mathbf{Q}} F$, a power of C with the regular F-action. The algebra $E' \otimes_{\mathbf{Q}} F = \prod_{\rho \in S} F$ acts on this power. Let e_Δ be the idempotent supported on Δ and let Z_Δ be its image. At an embedding $\tau : F \hookrightarrow \overline{\mathbf{Q}}$, the F-eigenspace of $H_1(Z_\Delta)$ is

$$\bigoplus_{\rho \in \Delta} V_{\tau\rho}.$$

It therefore has dimension $2r$ and signature (r, r). Tensor a polarization on C with the positive rational form $(a, b) \mapsto \mathrm{Tr}_{F/\mathbf{Q}}(a\bar{b})$ and restrict to Z_Δ. The resulting polarization induces conjugation on F.

The map $v \mapsto e_\Delta(v \otimes 1)$ comes from a rational homomorphism $f_\Delta : C \to Z_\Delta$. Pullback of the determinant line at the identity embedding of F is the component indexed by Δ; the other embeddings give its conjugate components. Consequently the rational subspaces $f_\Delta^* W_F^r(Z_\Delta)$ span the rational Hodge classes under consideration. The spanning assertion descends from F to $\mathbf{Q}$, since it is an inclusion of rational vector spaces.

All the Z_Δ are factors, up to isogeny, of powers of C. After descending the finitely many maps to a common number field and extending the chosen place, these factors have good reduction by the Néron–Ogg–Shafarevich criterion. The rational homomorphisms specialize after clearing denominators. The comparison theorems and functoriality then specialize the displayed equality with its original rational coefficients. □

3 Weil classes and nonzero Lefschetz pairings

We first explain what a simultaneous lift of a Weil class and a divisor would achieve. In characteristic zero, a Weil class with a nonzero Lefschetz pairing already belongs to the divisor algebra. We then show that a nonzero pairing on the special fiber can be detected by a divisor associated with an invertible, conjugate-linear endomorphism. Section 4 will lift this endomorphism together with the Weil class.

3.1 A characteristic-zero observation

Lemma 3.1 (Weil classes and the Lefschetz projector)**.** *Let $A/\mathbf{C}$ have an E-action of rank $2s$ as above. Then precisely one of the following holds:*

$$W_E^s(A) \subset \mathrm{Lef}^s(A), \qquad \text{or} \qquad W_E^s(A) \perp \mathrm{Lef}^{g-s}(A).$$

In particular, a rational Weil class which has nonzero pairing with a complementary Lefschetz class is itself Lefschetz.

Proof. The perfect intersection pairing on the rational Lefschetz algebra defines a rational projection

$$P : H^{2s}(A, \mathbf{Q}(s)) \longrightarrow \mathrm{Lef}^s(A), \qquad \langle P\alpha, \lambda\rangle = \langle \alpha, \lambda\rangle \quad (\lambda \in \mathrm{Lef}^{g-s}(A)).$$

Its kernel is the orthogonal complement of $\mathrm{Lef}^{g-s}(A)$. Both its image and kernel are stable under e^* for $e \in E^\times$: the Lefschetz algebra is stable under rational endomorphisms, and the adjoint of e^* is $\deg(e)(e^{-1})^*$. Thus P is $E^\times$-equivariant, and hence equivariant for the rational torus $\mathrm{Res}_{E/\mathbf{Q}}\,\mathbf{G}_m$.

The weights of the Weil space are τ^{2s}, one for each embedding of E. Each occurs with multiplicity one in all of $H^{2s}(A, \mathbf{Q}(s))$: the corresponding weight space is exactly $\bigwedge^{2s} V_\tau^*(s)$. Consequently P preserves the Weil space. These weights form a single Galois orbit, so the Weil space is an irreducible rational representation of this torus. The restriction of the idempotent P is therefore either zero or the identity. This gives the two alternatives. □

3.2 Finding the endomorphism that must lift

Let $A_0/\overline{\mathbf{F}}_p$ be the good reduction of a balanced Weil variety A, and put $g = \dim A$. Write θ for its polarization class and † for Rosati. For the eigenspace calculations below, fix one cohomological realization and let V be its degree-one homology, the dual of H^1. After splitting E, write V_ξ for its ξ-eigenspace and ψ for the polarization form on V. Set

$$U = \{u \in \mathrm{End}^0(A_0) : u^\dagger = u, \quad ue = \bar{e}u \text{ for all } e \in E\}.$$

For $t^\dagger = t$, let D_t be the rational divisor whose alternating form is $\psi(x, ty)$.

Lemma 3.2 (Detection by an invertible endomorphism)**.** *Fix any of the cohomological realizations, allowing the coefficient extension used for crystalline specialization. Suppose a specialized Weil class γ_0 has nonzero pairing with a complementary Lefschetz class. Then there is an invertible $u \in U$ such that*

$$\mathrm{Tr}(\gamma_0 D_u^s \theta^{g-2s}) \neq 0.$$

Proof. Hard Lefschetz on the divisor algebra gives

$$\mathrm{Lef}^{g-s}(A_0) = \theta^{g-2s}\,\mathrm{Lef}^s(A_0).$$

The polarization identity expresses products of s rational divisor classes as rational linear combinations of powers D^s. Hence there is a rational divisor D_t with $\mathrm{Tr}(\gamma_0 D_t^s \theta^{g-2s}) \neq 0$.

After splitting E, let p_ξ be its eigenspace projectors. The operation

$$\Pi(t) = \sum_\xi p_{\bar\xi} t p_\xi$$

is Galois invariant, so defines a rational linear map on $\mathrm{End}^0(A_0)$. Since $p_\xi^\dagger = p_{\bar\xi}$, it maps Rosati-fixed endomorphisms to U. For $x, y \in V_\xi$ one has

$$\psi(x, \Pi(t)y) = \psi(x, ty).$$

Replacing D_t by $D_{\Pi(t)}$ preserves the whole pairing. Indeed, a pure component of γ_0 fills V_τ in degree $2s$. Polarization factors cannot then use V_τ or $V_{\bar\tau}$, because ψ pairs those two spaces. All $2s$ degrees on $V_{\bar\tau}$ must come from the s divisor factors; each factor therefore contributes exactly its restriction to that eigenspace. Those restrictions are unchanged by Π.

Thus $u = \Pi(t) \in U$ satisfies $\mathrm{Tr}(\gamma_0 D_u^s \theta^{g-2s}) \neq 0$. At least one pure component contributes nontrivially. For its conjugate embedding $\bar\tau$, this forces $D_u^s|_{V_{\bar\tau}} \neq 0$, so $\psi(x, uy)|_{V_{\bar\tau}}$ is nondegenerate.

It remains to propagate invertibility from this eigenspace to all of them. Rosati positivity makes u semisimple [Mil86, Theorem 17.3], so $\ker(u^2) = \ker(u)$. Thus $v = u^2$, which commutes with E, is invertible on $V_{\bar\tau}$. Its E-characteristic polynomial belongs to $E[T]$. Indeed, the rationality of the ordinary characteristic polynomial of every rational endomorphism makes $e \mapsto \mathrm{Tr}(ev^n)$ a rational linear functional on E. The trace pairing expresses it uniquely as $e \mapsto \mathrm{Tr}_{E/\mathbf{Q}}(ea_n)$ for some $a_n \in E$. After splitting E, this says $\mathrm{Tr}(v_\xi^n) = \xi(a_n)$; Newton identities give the asserted characteristic polynomial. Its constant term has nonzero $\bar\tau$-image and is therefore nonzero in E, and under every embedding. Hence v, and consequently u, is invertible. □

The characteristic-polynomial rationality used here includes crystalline cohomology: for an integral endomorphism a, $\det(n - a \mid H^1) = \deg([n] - a)$ for all but finitely many integers n. Interpolation gives the same rational polynomial in every realization; clearing denominators treats rational endomorphisms [Mil99, Appendix, Proposition A.2].

4 Simultaneous lifting through a reductive subgroup

We now lift the Weil class together with the endomorphism u found in Lemma 3.2. The local task is to choose a balanced Hodge filtration preserving E and u. Their centralizer has symplectic derived factors and one similitude character, so the standard nonemptiness criterion supplies this filtration. Kisin–Zhou's lifting argument then realizes it on an abelian variety up to isogeny. Write σ for the Frobenius of K_0.

4.1 The general lifting principle

For a connected reductive group $H/\mathbf{Q}_p$, write $B(H)$ for its set of isocrystal classes. A geometric conjugacy class $\{\mu_H\}$ of cocharacters defines the subset

$$B(H,\{\mu_H\}) = \{[b_H] :\ \kappa_H(b_H) = \mu_H^\natural, \quad \nu_{b_H} \preceq \mu_H^\diamond\}.$$

Here $\Gamma = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, $\kappa_H(b_H)$ is the Kottwitz invariant in the coinvariants $\pi_1(H)_\Gamma$ of the algebraic fundamental group, and $\mu_H^\natural$ is the image of μ_H in this group. The dominant Newton point is ν_{b_H}, and $\mu_H^\diamond$ is the Galois average of the dominant representative of $\{\mu_H\}$. We say that a filtered isocrystal with tensors has *finite local-field descent* if it comes by scalar extension from such data over $L_0 \subset L$, where $L/\mathbf{Q}_p$ is finite and L_0 is its maximal unramified subfield.

Proposition 4.1 (Lifting additional tensors)**.** *Let $H \hookrightarrow G$ be connected reductive groups over $\mathbf{Q}_p$. Suppose a G-isocrystal represented by $[b] \in B(G)$ has a specified reduction to $[b_H] \in B(H)$. Let $\{\mu_H\}$ map to the desired G-conjugacy class $\{\mu\}$, and assume*

$$[b_H] \in B(H,\{\mu_H\}).$$

Over a finite extension of K_0 there is a weakly admissible filtered isocrystal with H-structure, underlying isocrystal b_H, and Hodge type $\{\mu_H\}$. Extending its structure group gives a filtered lift of b of type $\{\mu\}$.

If the isocrystal and tensor data descend to a finite unramified extension of $\mathbf{Q}_p$, the filtration can be chosen with finite local-field descent. The resulting filtered isocrystal is then admissible.

Proof. The weakly admissible period domain

$$\mathcal{F}(H, b_H, \{\mu_H\})^{\mathrm{wa}}$$

is nonempty by [DOR10, Theorem 9.5.10], or [RV14, Proposition 3.1]. A classical point is defined over a finite extension of K_0 and gives the required filtration. If the tensors have finite descent, the constrained flag variety is defined over a finite extension of $\mathbf{Q}_p$. Its $\overline{\mathbf{Q}}_p$-points are dense in its C_w-points. Since the weakly admissible locus is open, such a point can be chosen in the weakly admissible locus. It is defined over a finite extension $L/\mathbf{Q}_p$. Weak admissibility restricts to the descended Frobenius subobjects over L_0, and [CF00, Theorem A] makes the descended filtered isocrystal admissible. Scalar extension gives the assertion over K_0L. □

Tensors fixed by H become filtered morphisms, so endomorphisms defining H preserve the resulting filtration. We next identify the centralizer of E and u and check this nonemptiness criterion for its isocrystal.

4.2 The symplectic centralizer

We apply Proposition 4.1 to the centralizer of the endomorphisms we want to lift. Let $A_0/\overline{\mathbf{F}}_p$ have an action of a CM field E and a polarization whose Rosati involution induces complex conjugation on E. Put

$$D = H^1_{\mathrm{cris}}(A_0/W(\overline{\mathbf{F}}_p))[1/p], \qquad \mathrm{rank}_{E\otimes K_0} D = 2s.$$

Write β for the pairing on D dual to the homological polarization, with values in the isocrystal line of slope one. Suppose that $u \in \mathrm{End}^0(A_0)$ is invertible and satisfies

$$u^\dagger = u, \qquad ue = \bar{e}u \quad (e \in E). \tag{2}$$

All these data descend to a finite field.

Lemma 4.2 (The centralizer of E and u)**.** *There is a $\mathbf{Q}_p$-space (V_p, ψ_p) with $E \otimes \mathbf{Q}_p$ and u actions whose scalar extension to K_0 is isomorphic to (D, β, E, u) after forgetting Frobenius and trivializing the pairing line. For*

$$H = Z_{\mathrm{GSp}_E(V_p,\psi_p)}(u),$$

the group H is connected reductive. For suitable positive integers m_a, its geometric form is

$$H_{\overline{\mathbf{Q}}_p} \simeq \left\{ (g_a) \in \prod_a \mathrm{GSp}_{2m_a} : c(g_a) \text{ is independent of } a \right\}. \tag{3}$$

In particular,

$$H^{\mathrm{der}}_{\overline{\mathbf{Q}}_p} \simeq \prod_a \mathrm{Sp}_{2m_a}, \qquad H^{\mathrm{ab}} \simeq \mathbf{G}_m, \qquad \pi_1(H)_\Gamma \simeq \mathbf{Z},$$

with the last two identifications induced by the common multiplier.

Proof. Positivity of Rosati makes left multiplication by u self-adjoint for a positive definite trace pairing on $\mathrm{End}^0(A_0) \otimes \mathbf{R}$ [Mil86, Theorem 17.3]; hence u and u^2 are semisimple. Over an algebraic closure, decompose D simultaneously by the E-embeddings and the eigenvalues of u^2:

$$D = \bigoplus_{\tau,z} M_{\tau,z}.$$

The operator u identifies $M_{\tau,z}$ with $M_{\bar{\tau},z}$. On $M_{\tau,z}$ the form

$$B_{\tau,z}(x, y) = \beta(x, uy)$$

is alternating and nondegenerate. Thus $\dim M_{\tau,z} = 2m_{\tau,z}$. An automorphism commuting with E, u is determined by its action on one member of each conjugate pair; preserving β up to one scalar is exactly the common-similitude condition for these symplectic forms. This proves the geometric group description.

The $\mathbf{Q}_p$-model is standard linear algebra for the algebraic endomorphism algebra $S = \mathbf{Q}[E, u]$ with its Rosati involution. Its factors are quaternion algebras with orthogonal involution over the factors of $\mathbf{Q}[E^+, u^2]$, where E^+ is the maximal totally real subfield of E. The rational character of its cohomological realization and the even symplectic multiplicities above give the required alternating $S \otimes \mathbf{Q}_p$-module, by the usual description of modules over algebras with involution. Since the data have finite descent, the identification over K_0 may also be chosen with finite descent.

Finally the kernel of the common multiplier in (3) is a product of simply connected symplectic groups. It is the derived group, and the multiplier identifies the abelianization with $\mathbf{G}_m$. □

Theorem 4.3 (A filtration preserving E and u)**.** *Under (2), there is a finite extension K/K_0 and a weakly admissible filtration on $D \otimes_{K_0} K$, with finite local-field descent, such that its subspace $L = \mathrm{Fil}^1$ is E- and u-stable, is Lagrangian for β, and satisfies*

$$\dim L_\tau = s$$

at every embedding of E, after extending scalars to split E.

Proof. Choose the $\mathbf{Q}_p$-model of Lemma 4.2 and write Frobenius as $b\sigma$. It commutes with E, u and scales the pairing by p, so $b \in H(K_0)$ and $c(b) = p$.

Let $\{\mu_H\}$ be the class which is Siegel on each factor in (3): its weights are $1^{m_a}, 0^{m_a}$ and its common multiplier is t. This geometric class is Galois invariant. The multiplier detects its entire Kottwitz invariant, and hence

$$\kappa_H(b) = 1 = \mu_H^\natural.$$

Every slope of D lies in $[0, 1]$. On each symplectic factor the Newton weights have the form

$$a_1, \dots, a_m, 1 - a_m, \dots, 1 - a_1, \qquad 1 \geq a_1 \geq \cdots \geq a_m \geq \tfrac{1}{2}.$$

Thus $\sum_{i=1}^{j} a_i \leq j$ for $1 \leq j \leq m$, which is exactly the dominance inequality below the Siegel cocharacter in type C_m. Consequently

$$\nu_b \preceq \mu_H^\diamond = \mu_H, \qquad [b] \in B(H, \{\mu_H\}).$$

Proposition 4.1 now supplies the filtration. Its defining cocharacter commutes with E, u, and the weight-one space is Lagrangian. Since $\sum_z m_{\tau,z} = s$, it has dimension s in each E-eigenspace.

Finally, a symplectic similitude on $M_{\tau,z}$ has determinant $c^{m_{\tau,z}}$. Therefore every element of H satisfies

$$\det_E(g) = c(g)^s. \tag{4}$$

Thus the cocharacter fixes the Tate-twisted Weil tensors and has the balanced multiplicities required for a lift of Weil type. □

4.3 From the filtration to a marked abelian lift

We now use Kisin–Zhou to realize the filtration on an abelian variety, while preserving the Weil class and the endomorphism. The quasi-isogeny in the following statement specifies how their specializations are identified with the original data.

Corollary 4.4 (Simultaneous lifting of the Weil class and the divisor)**.** *Let $A/\overline{\mathbf{Q}}$ be a polarized abelian variety of balanced Weil type for E, with good reduction A_0. Suppose an invertible $u \in \mathrm{End}^0(A_0)$ satisfies (2). For every rational Weil class $\gamma \in W_E^s(A)$, there exist an abelian variety C over a local characteristic-zero field, a quasi-isogeny $f_0 : A_0 \longrightarrow C_0$, and a rational Weil class $\widetilde{\gamma} \in W_E^s(C)$ such that:*

1. *the E-action, the transported polarization, and $f_0 u f_0^{-1}$ lift to C;*
2. *$f_0^* \widetilde{\gamma}_0 = \gamma_0$ simultaneously in all the realizations under consideration;*
3. *the rational divisor D_u lifts through f_0 to a rational divisor $\widetilde{D}_u$ on C.*

Hodge classes on C are interpreted through a complex embedding; their indicated specializations are supplied by the canonical tensors of the Shimura construction.

Proof. To retain the Weil class, we use the following group on rational Betti homology:

$$G_W = \{g \in \mathrm{GSp}_E(H_1(A), \psi) : \det_E(g) = c(g)^s\}.$$

Here ψ is the polarization form and c its similitude character. The filtration orbit determined by A consists of the E-stable Lagrangians having dimension s in every eigenspace. The kernel of the

multiplier in G_W is geometrically a product of SL_{2s}, and its real adjoint factors have signature (s,s); hence the given Hodge homomorphism defines a Shimura datum of Hodge type. Each rational Weil class, with its Tate twist, is a G_W-invariant tensor.

Theorem 4.3 gives an admissible filtration in this orbit. Apply the prescribed-filtration alternative of Proposition A.1. We obtain B, a quasi-isogeny $a : A_0^n \to B_0$, and a rational Hodge class κ with

$$(aj)^*\kappa_0 = \gamma_0, \qquad a^* \operatorname{Fil}^1 H^1_{\mathrm{dR}}(B) = L^{\oplus n},$$

where j includes the first factor.

Let $\mathrm{pr}_1 : A_0^n \to A_0$ be the first projection. The rational projector $R_0 = aj \,\mathrm{pr}_1\, a^{-1}$ cuts out the first factor through the marking. The operators acting by E and u on this factor and by zero on the others also transport to rational endomorphisms of B_0. They all preserve the filtration and therefore lift uniquely to rational endomorphisms of B, by the usual Grothendieck–Messing lifting criterion.[1] Uniqueness preserves their algebraic relations. Let C be the image of the lifted projector, in the isogeny category. The marking aj identifies A_0 with C_0 up to isogeny and gives f_0. Restrict κ to C to obtain $\widetilde{\gamma}$. Its specializations are those of γ under this marking. It is a Weil class: the determinant summand is defined functorially from the E-action, and this condition can be checked in crystalline cohomology and carried back through comparison.

The polarization homomorphism is filtered because L is Lagrangian, and u is filtered because $uL = L$. The polarization therefore lifts; its rational divisor class is ample since its specialization is ample. Their composite, the symmetric homomorphism defining D_u, also satisfies the lifting criterion and yields $\widetilde{D}_u$. This proves all assertions with the required marking. □

5 The Weil dichotomy and the main theorem

The simultaneous lift of Section 4 allows us to apply the characteristic-zero Lefschetz projector to a specialized Weil class with nonzero pairing. This gives the dichotomy from which the rationality theorem follows.

Theorem 5.1 (The specialized Weil dichotomy)**.** *Let γ be a rational Weil class on a balanced Weil variety $A/\overline{\mathbf{Q}}$ with good reduction A_0. Either all complementary Lefschetz pairings of γ_0 vanish in every realization, or γ_0 is a simultaneous rational Lefschetz class.*

Proof. Put $g = \dim A$ and let θ be the specialized polarization. If some pairing is nonzero in some realization, choose u by Lemma 3.2. Apply Corollary 4.4 to γ, E, u, and the polarization. It supplies, up to isogeny, an abelian lift B with a rational Weil class η and rational divisors D, Θ whose specializations pull back to γ_0, D_u, θ, respectively, under the specified special-fiber marking. All these compatibilities hold simultaneously in the realizations under consideration.

Choose a complex realization of B. Compatibility with the chosen nonzero specialized pairing gives

$$\mathrm{Tr}(\eta D^s \Theta^{g-2s}) \neq 0.$$

Lemma 3.1 implies that η is a rational Lefschetz class. Write it as a rational polynomial in divisor classes on B. The same identity specializes in every realization, because its terms are absolute Hodge classes. Pulling back by the special-fiber marking expresses γ_0 as one rational polynomial in divisors on A_0, simultaneously in all realizations. □

[1] In the isogeny category, use Drinfeld's lifting modulo p^2 [Kat81, Lemma 1.1.3], followed by Grothendieck–Messing and formal existence; see [Ill15, Section 4.2].

For these specialized Weil classes, the rational Lefschetz projection of Proposition 2.3 is thus either zero or the class itself.

Remark 5.2 (The uniform form). The dichotomy can also be stated uniformly for the entire Weil space, using the endomorphism space U defined in Section 3.2. If U has no invertible element, Lemma 3.2 makes every pairing vanish. If U has an invertible element, fix any rational Weil class and apply Corollary 4.4 using this element. Its divisor on the lift is nondegenerate on every E-eigenspace, so $D^s\Theta^{g-2s}$ has nonzero pairing with at least one determinant line. It therefore pairs nontrivially with the rational Weil space. Lemma 3.1 puts that entire space in the Lefschetz algebra. Transport and specialization give the uniform second alternative for the original specialized Weil classes.

5.1 Deduction of the rationality theorem

Proof of Theorem 1.1. Classes of codimension zero or top codimension have the assertion by the cycle normalization of the trace. In the other degrees, apply the CM transport theorem, Proposition 2.4, to obtain a CM abelian variety $C/\overline{\mathbf{Q}}$, a rational Hodge class κ on C, and a rational homomorphism of special fibers $f_0 : A_0 \to C_0$ such that

$$\gamma_{0,v} = f_0^*\kappa_{0,v} \qquad \text{for every } v.$$

Proposition 2.5 expresses κ as a sum of rational pullbacks of balanced Weil classes. For each such class, the dichotomy of Theorem 5.1 gives the rational pairing property: in the orthogonal branch every value is zero; in the other branch every value is the degree of one rational divisor intersection. Lemma 2.2 preserves this property through both reductions and proves the pairing assertion. Proposition 2.3 then gives the unique common rational Lefschetz projection and the orthogonal decomposition asserted in Theorem 1.1. □

6 Consequences for specialization

6.1 A theory of rational Tate classes

Milne's theory of rational Tate classes asks for rational structures on Tate cohomology that are compatible across realizations and with the usual cycle operations. The rationality theorem supplies such a structure containing the reductions of Hodge classes.

Let $\mathcal{S}_0$ be the class of k-varieties whose connected components are products of abelian varieties and projective spaces. By [Mil09, Theorem 4.5], the CM case of Theorem 1.1 gives a good theory $\mathcal{R}$ of rational Tate classes on $\mathcal{S}_0$. Its realization maps are injective, it is stable under pullback, and it contains the specializations of rational Hodge classes on CM abelian varieties. The same specialization property holds without the CM hypothesis.

Corollary 6.1. *For every $A/\overline{\mathbf{Q}}$ with good reduction A_0 and every rational Hodge class γ of codimension r, there is a unique $\rho_\gamma \in \mathcal{R}^r(A_0)$ such that*

$$(\rho_\gamma)_\ell = \gamma_{0,\ell} \quad (\ell \neq p), \qquad (\rho_\gamma)_p \otimes 1 = \gamma_{0,p}.$$

Proof. For the unit and top-degree classes use their rational algebraic representatives. Otherwise CM transport gives $\gamma_0 = f_0^*\kappa_0$ with κ a CM Hodge class. The goodness axiom supplies $\rho_\kappa \in \mathcal{R}^r(C_0)$. Choose $N > 0$ with Nf_0 an actual homomorphism and set

$$\rho_\gamma = N^{-2r}(Nf_0)^*\rho_\kappa.$$

Functoriality gives all the required components; injectivity of any one realization gives uniqueness. □

6.2 Algebraic specialization under the CM Hodge conjecture

Corollary 6.2. *Assume that every rational Hodge class on a CM abelian variety is algebraic. Under the hypotheses of Theorem 1.1, there is a single $z_\gamma \in \mathrm{CH}^r(A_0)_{\mathbf{Q}}$ such that*

$$\mathrm{cl}_\ell(z_\gamma) = \gamma_{0,\ell} \quad (\ell \neq p), \qquad \mathrm{cl}_{\mathrm{cris}}(z_\gamma) \otimes 1 = \gamma_{0,p}.$$

For every complementary rational algebraic cycle y, all the corresponding traces equal $\deg(z_\gamma \cdot y) \in \mathbf{Q}$.

Proof. Use CM transport to write $\gamma_0 = f_0^* \kappa_0$. By the assumed CM Hodge conjecture, κ is represented by a rational cycle Z on C. The representative can be taken over $\overline{\mathbf{Q}}$: a complex representative spreads in a Hilbert scheme of C, and specialization to an algebraic point on the same connected component preserves its Betti class. Specializing at the chosen place gives one cycle Z_0 whose realizations are κ_0. If $h = N f_0$ is an actual homomorphism, then

$$z_\gamma = N^{-2r} h^* Z_0$$

has the required realizations. Here we use the usual Chow pullback for a map between smooth varieties. Compatibility with intersection and trace gives the last assertion. □

A The two applications of Kisin–Zhou

The proof uses Kisin–Zhou's results in two ways: to pass to a CM lift in Section 2, and to realize a chosen admissible Hodge filtration in Section 4. In each case we need a quasi-isogeny identifying the specialization of the original Hodge class with that of the lifted class. The following proposition records both conclusions.

Proposition A.1 (Marked transport)**.** *Let $A/\overline{\mathbf{Q}}$ be a positive-dimensional polarized abelian variety with good reduction A_0. Put $V = H_1(A(\mathbf{C}), \mathbf{Q})$ and write ψ for its polarization form. Suppose $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), X_{\mathrm{Siegel}})$ is a Hodge embedding containing the Hodge point of A. Let $\gamma \in H^{2r}(A(\mathbf{C}), \mathbf{Q}(r))$ be a rational G-invariant tensor. There exist $n \geq 1$, an abelian variety B in characteristic zero with good reduction, a rational Hodge class κ on B, and a quasi-isogeny $a : A_0^n \to B_0$ such that*

$$(aj)^* \kappa_{0,v} = \gamma_{0,v} \qquad \text{for every } v, \tag{5}$$

where $j : A_0 \to A_0^n$ includes the first factor. One may require either:

(i) *$B/\overline{\mathbf{Q}}$ is CM;*

(ii) *for a specified weakly admissible filtration F on $D_A = H^1_{\mathrm{cris}}(A_0/W(k))[1/p]$ in the original G-filtration orbit,*

$$a^* \operatorname{Fil}^1 H^1_{\mathrm{dR}}(B/K) = (F^1)^{\oplus n} \tag{6}$$

over a finite extension K/K_0. Here the filtered isocrystal and its tensors are assumed to descend to a finite extension $L/\mathbf{Q}_p$, with underlying isocrystal over its maximal unramified subfield L_0.

In (ii)*, Betti classes are interpreted through a complex embedding. Their specializations are those of the canonical Shimura tensors. The assertion holds simultaneously for a finite family of classes* γ*.*

Proof. The group $G_{\mathbf{Q}_p}$ need not be quasi-split, as required for the form of Kisin–Zhou's lifting theorem used here. We first arrange this hypothesis by restriction of scalars. Choose a totally real field P whose completions at p split $G_{\mathbf{Q}_p}$, and set

$$J = (\mathrm{Res}_{P/\mathbf{Q}}\, G_P) \underset{\mathrm{Res}_{P/\mathbf{Q}}\, \mathbf{G}_m}{\times} \mathbf{G}_m,$$

where the last map is diagonal. Thus J is the restriction of scalars with one common similitude multiplier, and $J_{\mathbf{Q}_p}$ is quasi-split. The trace polarization and diagonal Hodge homomorphism give its Hodge-type datum. By [KZ25, Propositions 5.3.8 and 2.1.9], it admits the strongly admissible integral model needed for their lifting results, including at $p = 2$, with Hodge representation

$$W \simeq (V \otimes_{\mathbf{Q}} P)^{\oplus m}.$$

Put $n = m[P : \mathbf{Q}]$. At the diagonal Hodge point the corresponding abelian variety C admits a quasi-isogeny $f : A^n \to C$. After finite extension this point extends to the integral model; write x for its reduction and $f_0 : A_0^n \to C_x$ for the specialized quasi-isogeny.

Choose a basis of P beginning with 1, so the first factor of A^n maps to $v \mapsto (v \otimes 1, 0, \dots, 0)$ in W. Extend the alternating form underlying γ by the normalized trace

$$\eta = [P : \mathbf{Q}]^{-1} \,\mathrm{Tr}_{P/\mathbf{Q}}(\gamma_P) \circ \mathrm{pr}_1^{\otimes 2r}\,.$$

With its Tate twist, this is a rational J-invariant tensor, and $j^* f^* \eta_C = \gamma$. Use the polarization to remove the Tate twist and include the resulting tensor among the structure tensors.

For (i), [KZ25, Theorem 2.2.7] gives a CM lift B of a point in the tensor isogeny class of x, with marking $b : C_x \to B_0$.

For (ii), transport $F^{\oplus n}$ to D_x using f_0. The diagonal map $G \to J$ puts it in the required cocharacter orbit; finite descent and Colmez–Fontaine make it admissible [CF00]. The realization argument in the proof of [KZ25, Theorem 2.2.7, p. 14] applies to this filtration: the affine Deligne–Lusztig map of their Theorem 2.2.5 supplies the point in the isogeny class, and their Proposition 2.2.3 supplies its lift. Thus we obtain B and a marking $b : C_x \to B_0$ with the prescribed filtration.

In both cases the marking preserves the structure tensors in every realization. Set $a = bf_0$, and let $\lambda \in \mathbf{Q}^\times$ be the polarization multiplier of b. Tensor compatibility gives $(aj)^* \eta_{B,0,v} = \lambda^r \gamma_{0,v}$, so $\kappa = \lambda^{-r} \eta_B$ gives (5) in every realization. □

References

[And92] Y. André, *Une remarque à propos des cycles de Hodge de type CM*, in S. David (ed.), Séminaire de Théorie des Nombres, Paris, 1989–90, Progress in Mathematics 102, Birkhäuser, 1992, pp. 1–7.

[Bla94] D. Blasius, *A p-adic property of Hodge classes on abelian varieties*, in U. Jannsen, S. Kleiman, and J.-P. Serre (eds.), Motives, Part 2, Proceedings of Symposia in Pure Mathematics 55, American Mathematical Society, 1994, pp. 293–308.

[CF00] P. Colmez and J.-M. Fontaine, *Construction des représentations p-adiques semi-stables*, Inventiones Mathematicae 140 (2000), 1–43.

[Del82] P. Deligne, *Hodge cycles on abelian varieties* (notes by J. S. Milne), in P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Mathematics 900, Springer, 1982, pp. 9–100.

[DOR10] J.-F. Dat, S. Orlik, and M. Rapoport, *Period Domains over Finite and p-adic Fields*, Cambridge Tracts in Mathematics 183, Cambridge University Press, 2010.

[Ill15] L. Illusie, *Grothendieck at Pisa: crystals and Barsotti–Tate groups*, in U. Zannier (ed.), Colloquium De Giorgi 2013 and 2014, Publications of the Scuola Normale Superiore 5, Edizioni della Normale, 2015, pp. 79–107.

[Kat81] N. M. Katz, *Serre–Tate local moduli*, in J. Giraud, L. Illusie, and M. Raynaud (eds.), Surfaces algébriques (Orsay, 1976–78), Lecture Notes in Mathematics 868, Springer, 1981, pp. 138–202.

[KZ25] M. Kisin and R. Zhou, *Strongly compatible systems associated to semistable abelian varieties*, preprint (2025), arXiv:2505.02165.

[Mil86] J. S. Milne, *Abelian varieties*, in G. Cornell and J. H. Silverman (eds.), Arithmetic Geometry, Springer, 1986, pp. 103–150.

[Mil99] J. S. Milne, *Lefschetz classes on abelian varieties*, Duke Mathematical Journal 96 (1999), 639–675; erratum to Theorem 5.9.

[Mil08] J. S. Milne, *The Tate conjecture over finite fields (AIM talk)*, notes, revised 2008.

[Mil09] J. S. Milne, *Rational Tate classes*, Moscow Mathematical Journal 9 (2009), 111–141.

[Mil20] J. S. Milne, *Hodge classes on abelian varieties*, preprint (2020), arXiv:2010.08857.

[Moo98] B. Moonen, *Models of Shimura varieties in mixed characteristics*, in A. J. Scholl and R. L. Taylor (eds.), Galois Representations in Arithmetic Algebraic Geometry, London Mathematical Society Lecture Note Series 254, Cambridge University Press, 1998, pp. 267–350.

[OAI26:Milne] OpenAI, *Milne's rationality conjecture for abelian varieties*, OpenAI Math Release preprint, 2026.

[RV14] M. Rapoport and E. Viehmann, *Towards a theory of local Shimura varieties*, Münster Journal of Mathematics 7 (2014), 273–326.